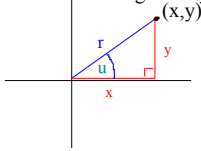


Welcome to Section 6.3 - Trigonometric Functions of Any Angle

Trigonometric Functions in the First Quadrant

If "u" is an angle in standard position with (x,y) being a point on the terminal side, then r = length of the segment connecting (0,0) to (x,y) as shown below.

Furthermore, by the Pythagorean Theorem, $r = \sqrt{x^2 + y^2}$.
Note that r = length of hypotenuse.



For angles in the first quadrant, we may define sine, cosine, and tangent in the following way:

$$\sin u = y/r, \quad \cos u = x/r, \quad \tan u = y/x$$
$$\csc u = r/y, \quad \sec u = r/x, \quad \cot u = x/y$$

Trigonometric Functions in ANY Quadrant

We will use the same definitions of sine, cosine, tangent, etc. to determine the values of the trig functions at angles in quadrants II, III, & IV. We will also use what is called a "reference angle" and "reference triangle". SEE NEXT PAGE . . .

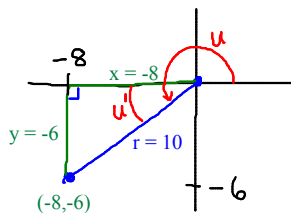
Page 1 2/5/01 1:00 PM

Trigonometric Functions in ANY Quadrant

To find the trigonometric function of any angle u, you will do the following:

1. Construct a "reference triangle". This is a right triangle with hypotenuse equal to the terminal side of the angle, and the base equal to the x-axis. The acute angle formed by the x-axis and the terminal side is called the "reference angle".
2. Fill in the side lengths of the reference angle, using the correct sign on the x & y values.
3. Use the definitions $\sin u = y/r$, $\cos u = x/r$, $\tan u = y/x$, $\csc u = r/y$, $\sec u = r/x$, $\cot u = x/y$

Example: An angle u has a terminal side with terminal point (-8, -6) as shown below. Find the exact values of $\sin u$, $\cos u$, and $\tan u$.



We construct a reference triangle that is a right triangle with the terminal side equal to the hypotenuse. We call the angle u' the reference angle. By the Pythagorean Theorem,

$$r = \sqrt{6^2 + 8^2} = 10$$

$$\sin u = y/r = -6/10$$

$$\cos u = x/r = -8/10$$

$$\tan u = y/x = -6/(-8) = 3/4$$

Notice that these values may be found by using SOHCAHTOA for the reference triangle.

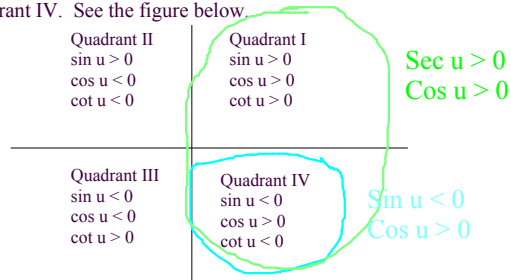
Page 2 2/5/01 1:39 PM

Example: If $\sec u > 0$ and $\cot u < 0$, in what quadrant does "u" lie in?

If $\sec u > 0$, $\cos u > 0$ since $\sec u = 1/(\cos u)$.

If $\cot u < 0$, then either $\cos u < 0$ and $\sin u > 0$ OR $\cos u > 0$ and $\sin u < 0$ since $\cot u = (\cos u)/(\sin u)$. The latter case MUST be true because $\cos u > 0$ by the first condition.

This means that $\cos u > 0$ and $\sin u < 0$. $\cos u > 0$ in quadrants I & IV. $\sin u < 0$ in quadrants III & IV. The common quadrant is IV so u is in quadrant IV. See the figure below.



Page 3 2/5/01 2:13 PM

Trigonometric Functions of Quadrant Angles

For the angles 90 deg, 180 deg, 270 deg, and 0 deg, and all coterminal angles, the trig functions are found with the same definitions as before where r = length of terminal side. We can not construct a reference triangle since the reference angle is either 0 degrees or 90 degrees. Note that "r" is always positive.

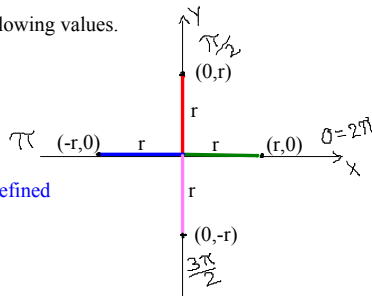
These quadrant angles have the following values.

$$\begin{aligned} \sin 0 &= y/r = 0/r = 0 \\ \cos 0 &= x/r = r/r = 1 \\ \tan 0 &= y/x = 0/1 = 0 \end{aligned}$$

$$\begin{aligned} \sin 90 &= \sin \pi/2 = y/r = r/r = 1 \\ \cos 90 &= \cos \pi/2 = x/r = 0/r = 0 \\ \tan 90 &= \tan \pi/2 = y/x = 1/0 = \text{undefined} \end{aligned}$$

$$\begin{aligned} \sin 180 &= \sin \pi = y/r = 0/r = 0 \\ \cos 180 &= \cos \pi = x/r = -r/r = -1 \\ \tan 180 &= \tan \pi = y/x = 0/-r = 0 \end{aligned}$$

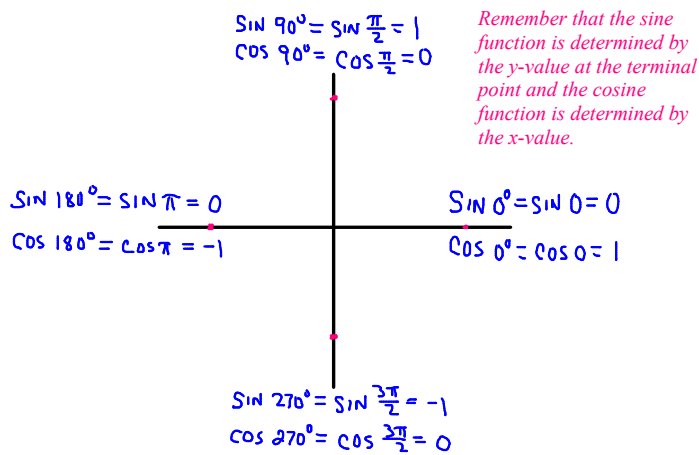
$$\begin{aligned} \sin 270 &= \sin 3\pi/2 = y/r = -r/r = -1 \\ \cos 270 &= \cos 3\pi/2 = x/r = 0/r = 0 \\ \tan 270 &= \tan 3\pi/2 = y/x = -r/0 = \text{undefined} \end{aligned}$$



The (x,y) coordinates of each terminal point are given above. The x & y values are equal to zero or +r or -r.

Page 4 2/5/01 2:31 PM

It's probably easiest just to memorize the following diagram.



Page 5 2/5/01 3:02 PM

Example: Find the values of the following functions of quadrant angles:

$\sec \frac{5\pi}{2}$, $\cot 7\pi$

$\sec 5\pi/2 = \sec \pi/2$ since $5\pi/2 = 2.5\pi = 2\pi + \pi/2$.

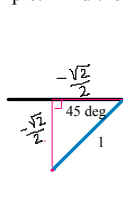
$\sec \pi/2 = 1/\cos \pi/2 = 1/0$ which is undefined.

$\cot 7\pi = \cot \pi$ since $7\pi = 6\pi + \pi$.

$\cot \pi = (\cos \pi)/(\sin \pi) = -1/0$ which is also undefined.

Note that we found the functions of the coterminal angles.

Example: Find the values of sine, cosine, and tangent of 225 deg.



Since 225 is 45 more than 180, the reference triangle for this angle is a 45-45-90 triangle with reference angle of 45 degrees and sides with lengths labeled as $(-\sqrt{2}/2)$ and $(-\sqrt{2}/2)$. Using

our definitions for sine, cosine, and tangent results in

$\sin 225 = y/r = \frac{-\sqrt{2}}{2}$ $\cos 225 = x/r = \frac{-\sqrt{2}}{2}$ $\tan 225 = y/x = 1$

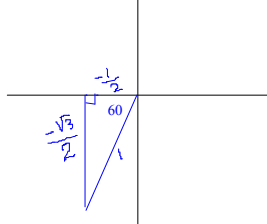
Page 6 2/5/01 3:11 PM

Finding Trigonometric Functions of Angles with Special-Case Reference Angles

In the last example, the angle 225 degrees corresponded to a reference angle that was 45 degrees and we were able to find the exact value of the trigonometric functions. We may find the trig functions of ANY angle that is a multiple of 30, 60, or 45 degrees.

Example: Find sine, cosine, and tangent of $\frac{22\pi}{3}$.

Since $22\pi/3 = 7\pi + \pi/3 = 6\pi + 4\pi/3$, this angle is coterminal with $4\pi/3$. $4\pi/3$ is equal to 240 degrees = 180 degrees + 60 degrees, so the reference angle is 60 deg and the reference triangle has sides labeled $-1/2$, $(-\sqrt{3})/2$, and 1.



Using the definitions for sine, cosine, and tangent results in

$$\sin(22\pi/3) = y/r = \frac{(-\sqrt{3}/2)}{1} = -\frac{\sqrt{3}}{2}$$

$$\cos(22\pi/3) = x/r = \frac{(-1/2)}{1} = -\frac{1}{2}$$

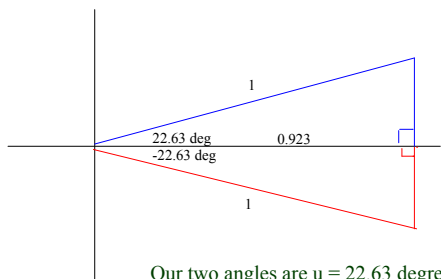
$$\tan(22\pi/3) = y/x = \frac{(-\sqrt{3}/2)}{(-1/2)} = \sqrt{3}$$

You should be able to find the EXACT value of ANY angle corresponding to a special case reference triangle!

Page 7 2/9/01 7:55 AM

Example: Use your calculator to find TWO values of "u" where "u" is between 0 and 360 degrees where $\cos u = 0.923$. Round "u" to 2 decimal places.

We may find the first value easily by using the inverse cosine function. We get $u = 22.63$ degrees, rounded to 2 places. Now we must think "What other angle has this same cosine value?"



The reference triangle for 22.63 deg has sides labeled $x = 0.923$ and $r = 1$ where $\cos u = x/r$.

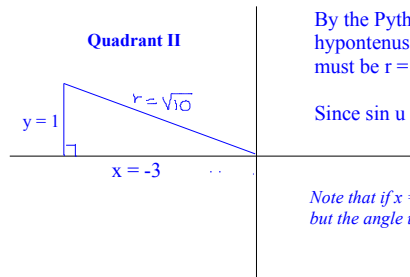
Another reference triangle with the same side lengths is the one for -22.63 degrees which is equal to $(360-22.63) = 337.37$ degrees.

Our two angles are $u = 22.63$ degrees and $u = 337.37$ degrees

Page 8 2/9/01 8:18 AM

Example: Given that $\cot u = -3$ and "u" is in quadrant II, find $\sin u$.

Since $\cot u = x/y$ and angle "u" is in quadrant II, we must construct a reference triangle with sides of length $x = -3$ and $y = 1$.



By the Pythagorean Theorem, the hypotenuse of the reference triangle must be $r = \sqrt{10}$.

Since $\sin u = y/r$, $\sin u = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$

Note that if $x = 3$ and $y = -1$, we get $\cot u = -3$ but the angle u is in quadrant IV.

Do the exercises for Section 6.3 and read the text!