Advanced Properties of Polynomials

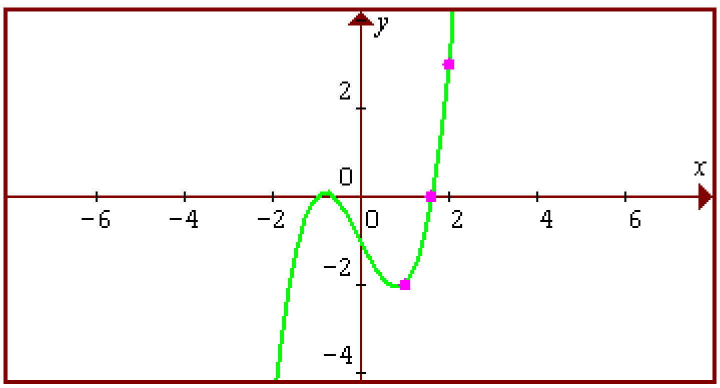
Intermediate Value Theorem: If f(x) is a polynomial, and f(a) ≠ f(b) for a<b, then f(x) takes on ever value from f(a) to f(b) in the closed interval [a,b].

Applied to polynomial zeros, The Intermediate Value Theorem states that if f(a) < 0 and f(b) > 0, then there must be a value x=c in the interval [a,b] such that f(c) = 0.

In other words, if the graph of a polynomial passes from negative to positive, it must pass through the x-axis at the value of a zero.

Example: Given $f(x) = x^3 – 2x - 1$, f(1) =- 2, and f(2) = 3, what does the Intermediate Value Theorem tell us about the existence of a zero?

f(x) must take on all values from y = - 2 to y = 3. Thus, it must take on the value y = 0. This means that there must be a zero at an x-value between x=1 and x=2. In fact, there is a zero, as shown below in the graph.



Fundamental Theorem of Algebra: If f(x) is a polynomial with degree n, then there is at least 1 complex zero x=c. Furthermore, if f(x) has degree n ≥ 1 with non-zero leading coefficient a_n, then

f(x) has exactly n linear factors and may be written as $f(x)= a_n(x - c_1)(x - c_2) . . . (x - c_n)$ where $c_1, c_2, c_3, ...,c_n$ are real or complex zeros and some of the zeros and associated factors may be repeated. The power on any repeated factor is known as its multiplicity. Factors that are not repeated have multiplicity = 1.

Example: If f(x) has degree 4 and zeros x=2, x=3, x=1, and x=0, write a general equation for f(x).

By the Fundamental Theorem of Algebra, f(x) will have 4 factors, each factor equal to $(x – c_i)$ where c_i is a zero. Since we are given 4 different zeros, f(x) must have 4 different factors. So f(x) would equal f(x) = **a**(x –2)(x – 3)(x – 1)(x – 0) where "**a**" is some coefficient.

Descartes' Rule of Signs: If f(x) is a polynomial with real coefficients with terms listed from highest to lowest power with k sign changes from term to term, then there will be "k" positive real zeros or "k - m" real zeros where m is some even integer.

Example: If $f(x) = x^4 - 3x^3 + 2x^2 + x - 1$, there are k=3 sign changes so there will be k=3 positive real zeros or k - m = 3 - 2 = 1 positive real zero.

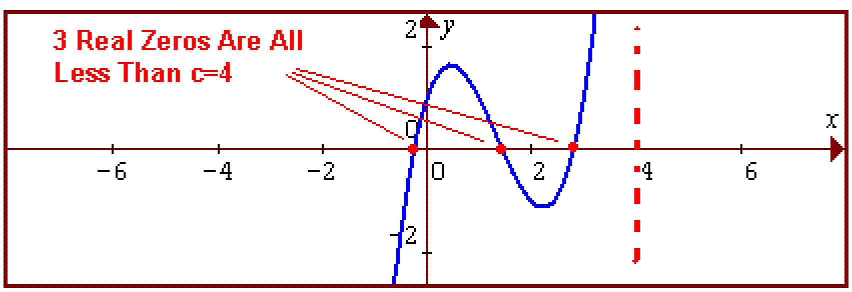
If f(x) is a polynomial with real coefficients with terms listed from highest to lowest power with k sign changes from term to term of f(-x), then there will be "k" negative real zeros or "k - m" real zeros where m is some even integer.

Example: If $f(x) = x^4 - 3x^3 + 2x^2 + x - 1$, then $f(-x) = x^4 + 3x^3 + 2x^2 - x - 1$ there are k=1 sign changes so there will be k=1 negative real zeros or k - m = 1 - 0 = 1 negative real zero. Here the only even integer less than 1 is m=0.

Note: Missing terms (coefficient = 0) are not considered. For example, $f(x) = -x^5 - 3x^2 - 2x^2 - 1$ would have zero sign changes.

Upper Bounds of Real Zeros: Let f(x) be a polynomial with real coefficients and leading coefficient *a* that is divided by (*x* - *c*) using synthetic division, where *c* > 0 and *c* is real.

c is an upper bound for all real zeros of f(x) if the leading coefficient *a* is positive provided none of the numbers in the bottom row of the synthetic division are negative. An example is shown below:



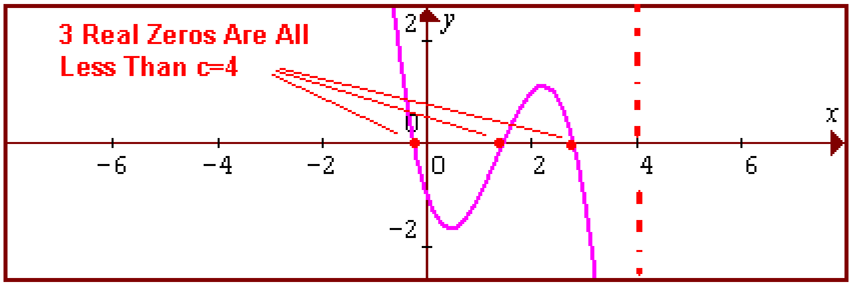
$f(x) = 1x^3 - 4x^2 + 3x + 1$, c = 4, a=1

Divide by (x - 4)

4	1	-4	3	1
		4	0	12
	1	0	3	13

No Negative Values

x = 4 is an upper bound for all real zeros *c* is an upper bound for all real zeros of f(x) if *c* > 0 and leading coefficient *a* is negative provided none of the numbers in the bottom row of the synthetic division are positive. An example is shown below:



$f(x) = -1x^3 + 4x^2 - 3x - 1$, $c = 4$, $a = -1$

Divide by $(x - 4)$

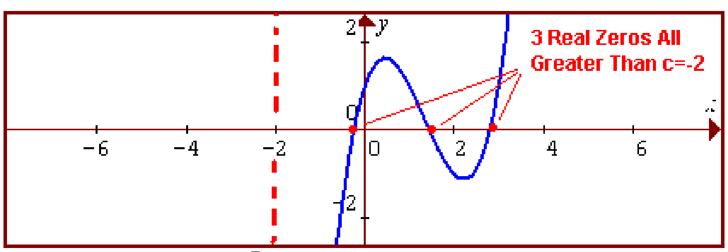
x = 4 is an upper bound for all real zeros

4	-1	4	-3	-1
		-4	0	-12
	-1	0	-3	-13

No Positive Values

Lower Bounds of Real Zeros: Let f(x) be a polynomial with real coefficients that is divided by (*x* - *c*) using synthetic division, where *c* < 0 and *c* is real.

c is a lower bound for all real zeros of f(x) if the bottom row of the synthetic division alternates in sign. If a value in the bottom row is zero, it can be considered to be positive or negative as needed to show the alternating pattern. An example is shown below:



$f(x) = 1x^3 - 4x^2 + 3x + 1$, $c = -2$

Divide by $(x - -2)$

X = -2 is a lower bound for all real zeros

-2	1	-4	3	1
		-2	12	-30
	1	-6	15	-29
	+	-	+	-

Alternating Values

The Rational Zero Test

If f(x) is a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \bullet\bullet\bullet + a_1 x + a_o$ with *integer* coefficients, then every rational zero of f(x) has the form

$$x = \frac{p}{q}$$

where p and q have no common factors other than q, and
p is a factor of the constant term a_o, and
and q is a factor of the leading coefficient a_n.

Example: Find all the possible rational zeros of $f(x) = 3x^3 + 2x - 2$

The possible zeros are $\pm \left\{ \frac{\text{Factors of -2}}{\text{Factors of 3}} \right\}$

This will be all combinations of +/- 1 or +/-2 divided by +/-1 or +/-3 which will be +/- { 1/1, 1/3, 2/1, 2/3} which will be {1, -1, 1/3, -1/3, 2, -2, 2/3, -2/3}.

Guess what? None of these are *actually* zeros, because the one real zero that exists is an irrational number. Remember: This test only gives the possible rational zero. The only guarantee is that if there is a rational zero, it will be contained in the list generated.

Complex Zeros Occur In Conjugate Pairs: If f(x) is a polynomial with real coefficients and has one complex zero x = a + bi, then x = a - bi will also be a zero. Furthermore, $x^2 - 2ax + a^2 + b^2$ will be a factor of f(x).

Example: Find the zeros of $f(x) = x^3 – x^2 – x – 2$. Then show that the complex zeros must occur in a conjugate pair.

If we use the Rational Zero Test, we find that the possible rational zeros are +/- {2, 1} or {2, -2, 1, -1}. Only x = 2 is an actual zero.

Dividing f(x) by (x-2) via synthetic division results in

2	1	-1	-1	-2
		2	2	2
	1	1	1	0

So $f(x) = (x – 2)(x^2 + x + 1)$
Letting $x^2 + x + 1 = 0$ results in quadratic solutions
$x = [-1 \pm \sqrt{(1 – 4)}] / 2$ or

$x = -1/2 + (\sqrt{3})/2 \bullet i$ and $x = -1/2 - (\sqrt{3})/2 \bullet i$

As you can see, when we solve the quadratic factors, we always get complex conjugate pairs.