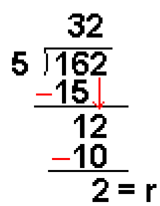
Factoring Polynomials

In order to factor a polynomial like $f(x) = x^3 + 2x - 3$, you first need to be able to divide the polynomial by a factor. Once you divide by a factor, you can rewrite f(x) as the product of your divisor times the quotient obtained. So first, we must review polynomial division.

Long Polynomial Division:

To divide a polynomial by another polynomial, you use the Division Algorithm in the same way you would divide $162 \div 5$. You Divide, Multiply, Subtract, and Carry Down, repeatedly as shown below.

Divide $162 \div 5 = 3$ (tens)
Multiply 3 (tens) • 5 = 30 • 5 = 15 (tens)
Subtract 16 (tens) – 15 (tens) = 1 (ten)
Carry Down 2



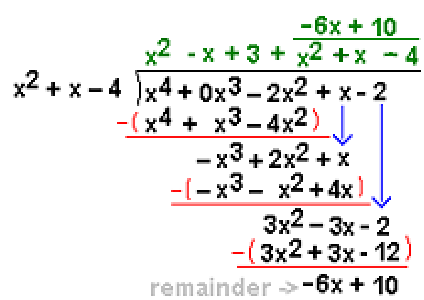
Divide $12 \div 5 = 2$
Multiply 2 • 5 = 10
Subtract 12 – 10 = 2
Nothing left to carry down, so we stop.
Our answer is 32 r 2 or 32 2/5

You can apply the same procedure to the division of two polynomials,
$(x^4 - 2x^2 + x - 2) \div (x^2 + x - 4)$

First, however, we must insert zero placeholders for missing terms and rewrite as
$(x^4 + 0x^3 - 2x^2 + x - 2) \div (x^2 + x - 4)$

Now, set up as a standard division problem and repeat the steps Divide, Multiply, Subtract, Carry Down over and over until the divisor no longer may be divided into the result at the bottom.

Step 1 – **Divide** leading term of dividend x^4 by leading term of divisor x^2. The result is x^2 and this is the first part of the answer.



Step 2 – **Multiply** your answer x^2 by divisor $(x^2 + x - 4)$ using the Distributive Property to get $x^4 + x^3 - 4x^2$. Place under dividend.

Step 3 – **Subtract** $x^4 + x^3 - 4x^2$ from divisor. Remember to correctly distribute the negative through the polynomial and add the opposite of each term!

Step 4 – **Carry Down** the x.

REPEAT these steps. **Divide** leading term at bottom $-x^3$ by leading term of divisor x^2 to get $-x$. This is the next part of your answer. **Multiply** your answer $-x$ by the divisor $(x^2 + x - 4)$ using the Distributive Property to get $-x^3 + 2x^2 + x$. Place result below dividend. **Subtract** to get $3x^2 - 3x$. **Carry Down** –2. **Divide** $3x^2$ by x^2 to get 3. **Multiply** 3 by $(x^2 + x - 4)$ and place below. **Subtract** to get $-6x + 10$. This is your remainder. You can write the remainder as a fraction as shown.

Synthetic Division - The Shortcut for Dividing by (x – c)
When dividing a polynomial f(x) by a linear factor (x-c), we can use a shorthand notation. saving steps and space. Here is the procedure:

Procedure For Synthetic Division of f(x) by (x – c):

1. Write the value of "c" and the coefficients of f(x) in a row. For example, if we divided f(x) = $3x^3 + 2x - 1$ by (x – **4**) we would write

4	3	0	2	-1

2. Carry down the first coefficient. In this case carry down the 3.

4	3	0	2	-1
	↓			
	3			

3. Multiply this carried down coefficient by the value of c.
In this case, multiply 3 • **4** = 12. Place this result in the next column.

4	3	0	2	-1
	↓	12		
	3			

4. Add the column entries and place result at bottom. In this case you add 0+12 to get 12. Multiply this addition result by "c" and place in next column. In this case you multiply 12 • **4** = 48.

4	3	0	2	-1
	↓	12	48	
	3	12		

5. Repeat Step 4 for all columns. In this example, you get

4	3	0	2	-1
	↓	12	48	200
	3	12	50	199

6. The bottom row of numbers reveals the answer along with the remainder. In this case, the numbers **3 12 50 199** indicate an answer of
3x^2 + **12**x + **50** r **199** or **3**x^2 + **12**x + **50** + **199**/(x – 4)

TIP: The answer will always have degree one less than the dividend. Always!

Also, when dividing by x plus something, c will be negative. For example, if you divide by (x + 5), this is the same as dividing by (x - - 5). So c = - 5.

Why does Synthetic Division work? If you compare long division side-by-side with synthetic division, you can fairly easily see why this shortcut works every time. This is left as a group exercise.

The Division Algorithm, Applied To Polynomials
If polynomial f(x) divided by polynomial D(x) results in quotient Q(x) with remainder R(x), then we may write f(x) = D(x)•Q(x) + R(x).

In other words, if we divide a polynomial by another polynomial, resulting in an answer, we can multiply that answer by our divisor, add the remainder, and we should get back our dividend.

Example: Divide f(x) = (x^2 –3x + 3) by (x –1). Then apply the Division Algorithm to rewrite f(x) as a product plus a remainder. Verify that the product with remainder added does indeed equal f(x).

The synthetic division for this results in

1	**1**	**-3**	**3**
		1	**-2**
	1	**-2**	**1**

which means f(x) ÷ (x – **1**) = **1x – 2** r **1**. So we can rewrite f(x) as
f(x) = (x^2 –3x + 3) = (x – **1**)**(x – 2)** + **1**

Verifying this, we multiply out (x – **1**)**(x – 2)** + **1** to get
x^2 –2x –x + 2 + 1 by the Distributive Property, which equals
x^2 – 3x + 3 after combining like terms.

NOTE: This is the same thing you did when you "checked your work" after dividing two numbers when first learning division. For example, if you divide 43 by 7 to get 6 r1, you checked your answer by multiply your quotient 6 by 7 and then adding the remainder 1 to get 6 • 7 + 1 = 43.

The Remainder Theorem
A factor, by definition, is a quantity that divides in evenly without remainder. The remainder theorem states If f(x) is divided by (x - c) with remainder r, then f(c) = r.

Where does this come from?
When we divide f(x) by (x-c) to get some quotient Q(x) and a remainder r, we then can, by the Division Algorithm write f(x) as

f(**x**) = (**x** – c) • Q(x) + r, where r will be a remainder of degree 0 since we divide by (x-c).
If **x** = **c**, we can write f(**c**) as
f(**c**) = (**c** – c) • Q(**c**) + r
= 0 • Q(**c**) + r
= 0 + r
= r

Example: If $f(x) = x^4 + 2x^3 – x^2 + x – 1$, find f(**2**) by dividing by (x – **2**). Then verify your result by directly evaluating f(**2**).

Synthetic Division results in

2	1	2	–1	1	–1
		2	8	14	30
	1	4	7	15	(29) Remainder

Since the remainder is 29, we can conclude that f(**2**) = 29.
If we substitute x = **2** into f(**x**), we get

$f(x) = 2^4 + 2(2)^3 – 2^2 + 2 – 1$
$= 16 + 16 – 4 + 2 - 1$
$= 29$

Zeros of Polynomials
A zero of a polynomial is a value x = a such that f(a) = 0. For example, **x** = **2** is a zero of $f(x) = x^2 + 4x – 8$ since $f(2) = 2^2 + 4 \bullet 2 – 8 = 0$.

Properties of Polynomial Zeros: If x = **a** is a zero of a polynomial f(x), then the following are all true:
- f(**a**) = 0
- (x – **a**) is a factor of f(x)
- The point (**a**, 0) is an x-intercept of the graph of f(x) if x= **a** is a real number.

The Factor Theorem: Using Division Results To Factor and Find Zeros
If f(x) divided by (x – c) results in zero remainder, then we can say that x = c is a zero. Also, we can say that (x – c) is a factor. This is justified by the Remainder Theorem and the Properties of Polynomial Zeros.

TIP: You can quickly check for zeros of a polynomial by synthetically dividing by (x – c). If the remainder is zero, then x=c is a zero and (x – c) is a factor. Furthermore, your synthetic division result gives you the factorization if x=c is a zero.

Example: Find a zero of $f(x) = x^3 + 3x^2 - 4x – 12$ by checking x =1, x = -1, and x=2 via synthetic division. Then use this result to write f(x) in factored form. Factor f(x) into a product of linear factors by factoring the quadratic factor.

Synthetic division by (x –1), (x + 1), and (x – 2) results in

1	1	3	-4	-12
		1	4	0
	1	4	0	-12

-1	1	3	-4	-12
		-1	2	2
	1	2	-2	-10

2	1	3	-4	-12
		2	10	12
	1	5	6	0

Only division by (x – **2**) results in r=0. Furthermore, as a result of the division, we can write $f(x) = (x – 2)(1x^2 + 5x + 6)$.

Now, factor $1x^2 + 5x + 6$ as (x + 2)(x + 3) and substitute in to get
f(x) = (x – **2**)(x + 2)(x + 3).

Note: When we divided by (x+1), we were dividing by (x- **-1**), so c = **-1** in the synthetic division.