

## Other Algebraic Equations

Including Factorable Higher Degree Polynomials, Rational Equations, Radical Equations, Equations With Rational Exponents, and Quadratic Form Equations

### Factorable Higher Degree Polynomials

A factorable higher degree polynomial is defined here as one that has a greatest common monomial factor that factors out in such a way that the resulting factors consist of the monomial and a polynomial of degree 2 or less, allowing easy solution.

Here is an example:  $3x^4 - 2x^3 + x^2 = 0$  may be factored as  $x^2(3x^2 - 2x + 1) = 0$  by applying the Distributive Property.

### Method For Solving These Factorable Higher Degree Polynomials

1. **Move all terms to one side of = leaving 0** on one side using the Addition Property of Equality.
2. **Factor out the greatest common factor** using the Distributive Property.
3. **Let each factor = 0 and solve each equation** using an appropriate method.

**Example: Solve  $2x^4 - 3x^3 = -x^2$**

$$2x^4 - 3x^3 = -x^2 \quad \text{Given}$$

$$2x^4 - 3x^3 + x^2 = 0 \quad \text{Add } x^2 \text{ to both sides using Addition Property of Equality}$$

$$x^2(2x^2 - 3x + 1) = 0 \quad \text{Factor out } x^2 \text{ using the Distributive Property}$$

$$\begin{array}{l} x^2 = 0 \text{ and} \\ 2x^2 - 3x + 1 = 0 \end{array} \quad \text{Apply the Zero Product Law}$$

$$\begin{array}{l} \text{Solve } x^2 = 0 \\ x = \pm\sqrt{0} \rightarrow x = 0 \end{array} \quad \text{Extract Square Roots}$$

$$\begin{array}{l} \text{Solve } 2x^2 - 3x + 1 = 0 \\ (2x - 1)(x - 1) = 0 \end{array} \quad \text{Factor using the Distributive Property}$$

$$x - 1 = 0 \quad \text{Apply the Zero Product Law}$$

$$2x - 1 = 0$$

$$x = 1 \text{ and } x = \frac{1}{2} \quad \text{Solve using Addition and Division Properties of Equality}$$

The three solutions are  $x = 0$ ,  $x = 1$ , and  $x = \frac{1}{2}$

*Remember to find all three solutions – many students forget the zero solution!*

## Equations Containing Rational Expressions

Equations containing rational expressions are here defined as those containing fractions with variables. For example,  $1/x + 1/3 = 1/(x+1)$  would be such an equation.

### Method For Solving Equations Containing Rational Expressions

1. **Identify the Least Common Denominator (LCD) of all the fractions.** In some instances you may need to factor the denominators first.
2. **Multiply both sides (all terms) by the LCD** using the Distributive Property. This will allow you to cancel all denominators and leave you with a fraction-less equation.
3. **Solve this equation using the appropriate method.**
4. **Check your solution.** Sometimes you will get a false solution even though you did all the work correctly!

Example: Solve  $\frac{2}{x-1} - \frac{1}{x} = \frac{3}{x^2-x}$

$$\frac{2}{x-1} - \frac{1}{x} = \frac{3}{x^2-x} \quad \text{Given}$$

$$\frac{2}{x-1} - \frac{1}{x} = \frac{3}{x(x-1)} \quad \text{We had to factor the denominator using the Distributive Property. Then determine the LCD of } x(x-1).$$

Now, multiply all terms on both sides by the LCD using the Distributive Property. Cancel out common factors.

$$\frac{x(x-1)}{1} \left( \frac{2}{x-1} - \frac{1}{x} \right) = \frac{x(x-1)}{1} \cdot \frac{3}{x(x-1)}$$

$$\frac{\cancel{x(x-1)}}{1} \cdot \frac{2}{\cancel{x-1}} - \frac{\cancel{x(x-1)}}{1} \cdot \frac{1}{x} = \frac{\cancel{x(x-1)}}{1} \cdot \frac{3}{\cancel{x(x-1)}}$$

$$2x - 1(x-1) = 3$$

$$2x - x + 1 = 3 \quad \text{Apply the Distributive Property and multiply out by } -1$$

$$x + 1 = 3 \quad \text{Combine Like Terms}$$

$$x = 2 \quad \text{Add } -1 \text{ to both sides using the Addition Property of Equality}$$

Does  $x=2$  check out in the original equation? If we plug it in, we get  $2 - \frac{1}{2} = \frac{3}{2}$  which is true. Thus,  **$x=2$  is the solution.**

## Equations Containing Radicals

Equations containing radicals are defined here as those containing the variable that we are solving for within a radical sign. For example, the equation

$$\sqrt{x^2 + 7} = x + 1 \text{ is a radical equation.}$$

### Method For Solving Equations Containing Radicals

1. **Isolate the radical on one side of equals** using Properties of Equality.
2. **Raise both sides of equation to a power that eliminates the radical.** If this is a square root equation, square both sides. If it is an nth root equation, raise both sides to the nth power.
3. **Solve this equation using the appropriate method.**
4. **Check your solution.** Many times you will get one or more false solutions, called *extraneous solutions*, even though you did all the work correctly!

**Example: Solve**  $\sqrt{x^2 - 1} + 3 = x$

$$\sqrt{x^2 - 1} + 3 = x \quad \text{Given}$$

$$\sqrt{x^2 - 1} = x - 3 \quad \text{Add } -3 \text{ to both sides using the Addition Property of equality}$$

$$(\sqrt{x^2 - 1})^2 = (x - 3)^2 \quad \text{Square both sides to eliminate the radical}$$

$$x^2 - 1 = (x - 3)(x - 3)$$

$$x^2 - 1 = x^2 - 6x + 9 \quad \text{Apply Distributive Property and combine like terms}$$

$$-10 = -6x \quad \text{Add } -x^2 \text{ and } -9 \text{ to both sides using Addition Property of Equality}$$

$$10/6 = 5/3 = x \quad \text{Divide both sides by } -6 \text{ using the Division Property of Equality}$$

**But, we must check this solution!**

$$\text{Does } \sqrt{\left(\frac{5}{3}\right)^2 - 1} + 3 = \frac{5}{3} ?$$

$$\sqrt{\frac{25}{9} - \frac{9}{9}} + 3 = \frac{5}{3}$$

$$\sqrt{\frac{16}{9}} + 3 = \frac{5}{3}$$

$$\frac{4}{3} + 3 = \frac{5}{3}$$

$$4 \frac{1}{3} \neq 1 \frac{2}{3}$$

The answer is **No Solution**

## Equations Containing Rational Exponents

Equations containing rational exponents are defined here as those with quantities raised to fractional powers on one side of equal and real numbers on the other such as the equation  $(x + 4)^{2/3} = 7$ . This type of equation is solved as follows:

### Method For Solving Equations Containing Quantities Raised To The M/N Power

1. **Isolate the quantity raised to the M/N power on one side of equals** using Properties of Equality.
2. **Raise both sides to the N/M power. If M of M/N is even, include a  $\pm$  sign on the numerical side. If M of M/N is odd, do *not* include a  $\pm$  sign.** The Multiplication Property of Exponents will cause the powers on your quantity to multiply to 1 and cancel.
3. **Solve using the appropriate method.**
4. **Check answers.** As with radical equations, it is possible to get extraneous solutions.

**Example: Solve  $(3 + 2x)^{2/5} + 3 = 7$**

$$(3 + 2x)^{2/5} + 3 = 7$$

Given

$$(3 + 2x)^{2/5} = 4$$

Add  $-3$  to both sides using Addition Property of Equality

$$[(3 + 2x)^{2/5}]^{5/2} = \pm 4^{5/2}$$

Raise both sides to  $5/2$  power. Include  $\pm$  sign.

$$3 + 2x = \pm 4^{5/2}$$

Exponent Rule results in exponents canceling

$$2x = -3 \pm 4^{5/2}$$

Add  $-3$  to both sides using Addition Property of Equality

$$x = (-3 \pm 4^{5/2}) / 2$$

Divide both sides by 2 using Division Property of Equality

$$x = (-3 \pm 32)/2$$

Simplify the power of 4

$$x = 29/2 \text{ or } x = -35/2$$

Now, you must check these answers in  $(3 + 2x)^{2/5} + 3 = 7$ !

$$\text{If } x = 29/2, (3 + 2(29/2))^{2/5} + 3 = (32)^{2/5} + 3 = 2^2 + 3 = 4 + 3 = 7$$

$$\text{If } x = -35/2, (3 + 2(-35/2))^{2/5} + 3 = (-32)^{2/5} + 3 = (-2)^2 + 3 = 4 + 3 = 7$$

Both answers check, so  $x = 29/2$  or  $x = -35/2$  are the solutions

### Why the $\pm$ sign in some cases and not in others?

When solving an equation with rational power with even numerator power,

like  $(x + 3)^{2/3} = 9$ , you are really solving the same equation as

$$(\sqrt[3]{x + 3})^2 = 9$$

To solve this equation, we extract square roots and then cube both sides. Extracting any even roots requires the use of a  $\pm$  sign to result in two possible answers.

Conversely, when solving an equation like  $(x + 1)^{3/2} = 8$ , we are solving the same equation as

$$(\sqrt{x + 1})^3 = 8$$

To solve this equation, we take the cube root of both sides and then square both sides to eliminate the radical. Extracting cube roots does not require a  $\pm$  sign. There will only one possible solution to this equation.

## Quadratic-Form Equations

Equations that may be written in the form  $ax^{2N} + bx^N + c = 0$  are known as quadratic form equations. For example, the equation  $2x^8 + 4x^4 + 2 = 0$  is a quadratic-form equation since the power of the first term, **8**, is twice the power of the second term **4**. The third term is a real number, 2. There are two methods for solving quadratic-form equations given below.

### Method One For Solving Quadratic Form Equations

1. **Move all terms to one side of =** by using the Addition Property of Equality.
2. **Factor the equation into 2 binomials, if possible** using the Distributive Property. If this is not possible, use Method Two.
3. **Let each factor = 0** as justified by the Zero Product Law.
4. **Solve each resulting equation using the appropriate method.**
5. **Check your solution** if radical equations or rational exponents were involved, you may get *extraneous solutions*.

**Example: Solve  $x^4 - x^2 = 12$**

$$x^4 - x^2 = 12$$

$$x^4 - x^2 - 12 = 0$$

$$(x^2 - 4)(x^2 + 3) = 0$$

$$x^2 - 4 = 0$$

$$x^2 + 3 = 0$$

$$x^2 = 4$$

$$x^2 = -3$$

$$x = \pm\sqrt{4}$$

$$x = \pm\sqrt{-3}$$

$$x = 2, x = -2$$

$$x = (\sqrt{3})i, x = -(\sqrt{3})i$$

**Given**

**Add -12 to both sides using Addition Property of Equality**

**Factor using Distributive Property**

**Let each factor = 0 by Zero Product Law**

**Add 4 to both sides using Addition Property of Equality**

**Add -3 to both sides using Addition Property of Equality**

**Extract Square Roots**

**Extract Square Roots**

**Simplify radicals**

**Simplify using definition of "i"**

Since no radicals nor rational exponents were involved, there are no extraneous solutions to check for.

### Method Two For Solving Quadratic Form Equations

1. **Move all terms to one side of =** by using the Addition Property of Equality.
2. **For the equation  $ax^{2N} + bx^N + c = 0$ , let  $u = x^N$ . This results in  $u^2 = x^{2N}$ . Make these substitutions into your equation.**
3. **Solve your equation  $au^2 + bu + c = 0$  for u** using an appropriate method.
4. **Let each solution for u be equal to  $x^N$ . Solve each of these equations for x.**
5. **Check your solutions** if radical equations or rational exponents were involved, you may get *extraneous solutions*.

**Example: Solve  $2x^4 = 5 + 3x^2$**

$$2x^4 = 5 + 3x^2$$

**Given**

$$2x^4 - 3x^2 - 5 = 0$$

**Move Terms to one side using Addition Property of Equality**

$$\text{Let } u = x^2, u^2 = x^4$$

**Make substitutions**

$$2u^2 - 3u - 5 = 0$$

**CONTINUED ON NEXT PAGE!**

$$u = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2(2)}$$

Solve for u using quadratic formula

$$u = \frac{3 \pm \sqrt{49}}{4}$$

$$u = 10/4 \text{ or } u = -1$$

$$x^2 = 10/4 \text{ and } x^2 = -1$$

Let  $x^2$  equal each of the solutions for u

$$x = \pm\sqrt{(10/4)}$$

Extract square roots. Then simplify radicals.

$$x = \pm\sqrt{(-1)}$$

Apply definition of "i".

$$x = (\sqrt{10})/2 \quad x = -(\sqrt{10})/2 \quad x = i \quad x = -i$$

*Note: We could have solved the quadratic by factoring*

*Remember to let each solution for u equal your power of x and solve for x – many students forget to do this!*