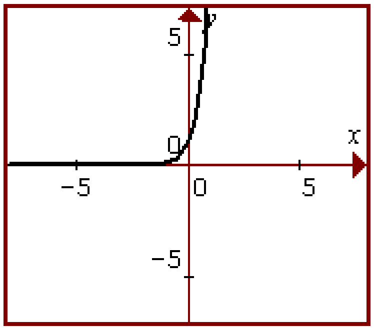
Logarithmic Functions

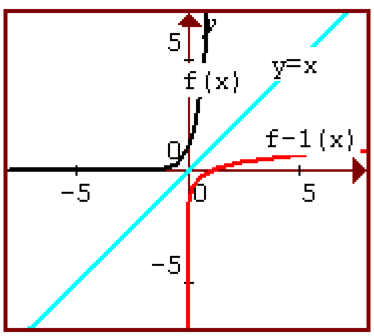
Logarithmic functions, or for short, log functions, serve as the inverse functions of exponential functions.

Why Do We Need Log Functions?

If you graph $f(x) = 10^X$, you get the graph as shown below.



We know that this function has an inverse for its entire domain since it passes the horizontal line test. In fact, if we switch x with y, and obtain solution points, we get the graph shown below, which represents a function. Also note the symmetry across the line y=x that is characteristic of a function graphed with its inverse.



So we know the inverse of $f(x) = 10^X$ exists, but how do we find it?

Using the procedure for finding the inverse, we write
$y = 10^X$
switch x with y to get
$x = 10^y$
And, now solve for y. But you can't!

So we define y as $y = LOG_{10}(x)$, so the inverse of f(x) = 10X is $f^{-1}(x) = LOG_{10}(x)$ which we pronounce as $f^{-1}(x)$ = the base10 log of x.

The BIG result here is that **$x = 10^y$ is equivalent to $y = LOG_{10}(x)$.**

Note: You may verify that $x = 10^y$ and $y = LOG_{10}(x)$ are equivalent to each other by showing that (x = 100, y=2) and (x=1000, y=3) are solutions for both statements. To evaluate the LOG function, you will have to use your LOG key in a scientific calculator.

The Log-Exponential Equivalence For ALL Bases
If "a" is a valid exponential base, then we can always say that

$x = a^y$ is equivalent to $y = LOG_a(x)$.

Mathematically, we say that *$x = a^y$ if-and-only-if $y = LOG_a(x)$*, which means

IF $x = a^y$, THEN $y = LOG_a(x)$ AND
IF $y = LOG_a(x)$, THEN $x = a^y$.

TIP: Just remember, you can always rewrite a simple exponential equation in equivalent log form and you can always rewrite a log form equation in exponential form.

Example: Rewrite $16 = 4^x$ in equivalent log form.
Here, the base is 4, so this will be a base-4 log. The exponent is x, and in the log statement, the exponent is always opposite the log with respect to the equal sign. So the log statement is **$LOG_4 16 = x$**.

Example: Rewrite $80 = 10^x$ in equivalent log form. Then use a scientific calculator to find x to 3 decimal places.
Here, the base is 10, so this will be a base-10 log. The exponent is x, and in the log statement, the exponent is always opposite the log with respect to the equal sign. So the log statement is

$LOG_{10}80 = x$. We find the base-10 LOG of 80 with a scientific calculator to be 1.903 (rounded), so x = 1.903 .

Example: Rewrite $20 = e^x$ in equivalent log form. Then use a scientific calculator to find x to 3 decimal places.
Here, the base is e, so this will be a base-e log. The exponent is x, and in the log statement, the exponent is always opposite the log with respect to the equal sign. So the log statement is

$LOG_e 20 = x$. We find the base-e LOG of 20 with a scientific calculator to be 2.996 (rounded), so x = 2.996 .

IMPORTANT NOTES: We refer to base-e logs as Natural Logs and we use a special notation of LN in place of LOG_e, so in the previous problem, you would denote $LOG_e 20 = x$ as LN(20) = x and use the LN key on your calculator. Your calculator typically only has keys for base-10 and base-e logs. For other bases, you can use a Change-of-Base formula discussed in the next section.

Example: $LOG_3 9$ = x in equivalent exponential form.
Since the base is 3, this will be a base-3 exponential statement. Also, since x is opposite the log statement, then x will be the exponent. This results in

$3^X = 9$

Example: LN(30) = x in equivalent exponential form.
We can rewrite this as $LOG_e 30 = x$. Since the base is e, this will be a base-e exponential statement. Also, since x is opposite the log statement, then x will be the exponent. This results in

$e^X = 30$

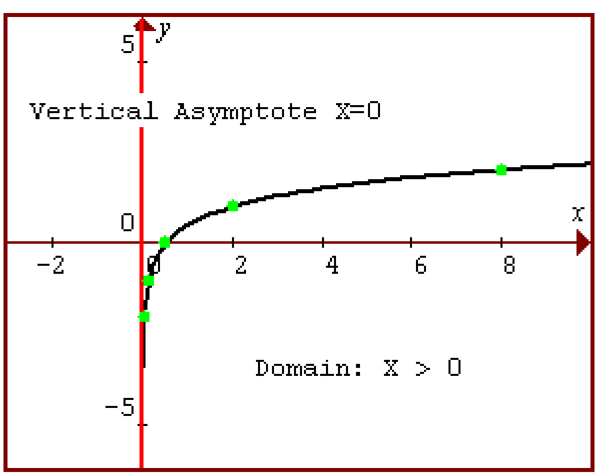
Graphing Log Functions
Since the log function is the inverse of the exponential, it will have a vertical asymptote instead of a horizontal asymptote. Also, since the range of the exponential function is restricted, the domain of the log function will be restricted. When you use the method for graphing shown below, both of these features will be fairly obvious.

Easy Method For Graphing Log Functions of The Form $y = LOG_a\, x$

1. Rewrite the log function in equivalent exponential form $a^y = x$.
2. Pick y-values first, calculate matching x-values, and plot points.
3. Draw a smooth curve through the points.

Example: Graph $f(x) = LOG_4 2x$
First, rewrite as $y = LOG_4 2x$.
Now, rewrite in the equivalent exponential form $4^y = 2x$, or solving for x, $(4^y)/2 = x$.

Choosing values of 0, 1, 2, -1, and –2 for y results in the solution points and graph shown below. One may see that x=0 is a vertical boundary for the graph and so x=0 is the vertical asymptote. The domain (possible x-values) is the set of all real numbers greater than zero. *Remember: The negative power means to invert and use the positive power. So if y=-2, $(4^{-2})/2 = (1/4)^2/2 = (1/16)/2 = 1/32$.*



$\frac{4^y}{2} = x$

x	y
1/2	0
2	1
8	2
1/8	-1
1/32	-2

Properties of Logarithms

- **LOG of 1 is 0**: Given the logarithmic function f(x) = $LOG_a x$, f(1) = 0.In other words, **$LOG_a 1 = 0$ for any legitimate exponential base a**.

- **LOG_a of a is 1**: Given the logarithmic function f(x) = $LOG_a x$, f(a) = 1.In other words, **$LOG_a a = 1$ for any legitimate exponential base a**.

- **Product Rule for Logs**: Given the logarithmic function f(x) = $LOG_a x$, f(UV) = f(U) + f(V).In other words, **$LOG_a(UV) = LOG_a U + LOG_a V$** for any legitimate exponential base a.

- **Quotient Rule for Logs**: Given the logarithmic function f(x) = $LOG_a x$, f(U/V) = f(U) - f(V).In other words,

  $$\text{Log}_a\left(\frac{U}{V}\right) = \text{Log}_a U - \text{Log}_a V$$

  for any legitimate exponential base a.

- **Power Rule for Logs**: Given the logarithmic function f(x) = $LOG_a x$, f(x^N) = N●f(x).In other words, **$LOG_a(x^N) = N \bullet LOG_a x$**.

- **Change of Base Rule for Logs**: Given the logarithmic function f(x) = $LOG_a x$, it is true, for any legitimate bases a and b, that

  $$\text{Log}_a x = \frac{\text{Log}_b x}{\text{Log}_b a}$$

- **Inverse Property**: Given the exponential function f(x) = a^x, the inverse of f(x) is the logarithmic function form is $f^{-1}(x) = LOG_a x$.

  Also, since $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$, we can say that

  $$a^{\log_a x} = x \quad \text{and}$$

  $$\log_a(a^x) = x$$

Example: Use properties of logs to rewrite 3LN(x) – (1/2)LN(2) + LN(z) by combining log terms into a single log.

3LN(x) – (1/2)LN(2) + LN(z)	
= $LN(x^3) - LN(2^{1/2}) + LN(z)$	Applying the **Power Rule For Logs** twice
= $LN(x^3) - LN(\sqrt{2}) + LN(z)$	Applying the rule for rational exponents
= $LN[(x^3)/(\sqrt{2})] + LN(z)$	Applying the **Quotient Rule For Logs**
= $LN[(x^3)/(\sqrt{2})] \bullet z]$	Applying the **Power Rule For Logs**
= $LN[(z \bullet x^3)/(\sqrt{2})]$	Multiply the fraction by z

Example: Use properties of logs to rewrite the expression shown below as a sum or difference of logs or multiples of logs.

$$LN\left(\frac{2x^3}{\sqrt{x}}\right)$$

$$LN\left(\frac{2x^3}{\sqrt{x}}\right)$$

$= LN(2x^3) - LN(\sqrt{x})$	Application of **Quotient Rule For Logs**
$= LN(2x^3) - LN(x^{1/2})$	Definition of rational exponents
$= LN(2) + LN(x3) - LN(x^{1/2})$	Application of **Product Rule For Logs**
$= LN(2) + 3LN(x) - \frac{1}{2} LN(x)$	Application of **Power Rule For Logs**

Example: Rewrite $3^X = 28$ in log form. Then rewrite this log as a ratio of two base-10 logs and also as two base-e logs. Then evaluate your ratios using a scientific calculator in order to solve for x.

The equivalent log form of $3^X = 28$ is $LOG_3 (28) = x$.

Now, applying the Change of Base Formula, we get

$$LOG_3(28) = \frac{LOG_{10}(28)}{LOG_{10}(3)}$$

and also $$LOG_3(28) = \frac{LN(28)}{LN(3)}$$

Using a scientific calculator, for either expression results in 3.033103256…

So, we conclude that x = 3.033, rounded to the thousandth, which is what we would expect, since $3^3 = 27$, and we are finding the power of 3 resulting in 28.