Applications of Exponential and Log Equations

Exponential and Logarithmic functions have perhaps more real-world applications than any other class of functions at the pre-calculus level and beyond. These functions govern population increase as well as interest income in a bank. And since (it seems) virtually everything decays exponentially, we can apply exponential decay equations to many different applications (see http://www.mathmotivation.com/science/carvalue.html)

Exponential Growth and Decline:

Exponential growth and decline is governed by the function: $\mathbf{N(t) = N_o e^{kt}}$

where

N_o = the original amount at time t=0.

k = the (decimal) rate of growth per time period, be it years, days, etc.

t = the units used in the time period.

N(t) = the amount at time t

NOTE: N(t) means "N as a function of t", NOT "N times t"!

Example: A population grows exponentially 3% per year. If the population is initially 1000, how many years will it take for the population to double to 2000? How many years will it take for the population to reach 4000?

Since the original amount is 1000, N_o=1000. Also, we know that N(t) = 2000 and also N(t) = 4000. In both cases, k = 0.03 . So we have to solve the equations:

$2000 = 1000e^{0.03t}$ and
$4000 = 1000e^{0.03t}$.

Here are the solutions:

$2000 = 1000e^{0.03t}$	Given
$2 = e^{0.03t}$	Division Property of Equality
$LN(2) = LN(e^{0.03t})$	If A = B, LN(A) = LN(B)
$LN(2) = 0.03t \bullet LN(e)$	Power Rule For Logs
$LN(2) = 0.03t$	$Log_a(a) = 1$
$[LN(2)]/0.03 = t \approx 23.1$ yrs	Division Property of Equality

$4000 = 1000e^{0.03t}$	Given
$4 = e^{0.03t}$	Division Property of Equality
$LN(4) = LN(e^{0.03t})$	If A = B, LN(A) = LN(B)
$LN(4) = 0.03t \bullet LN(e)$	Power Rule For Logs
$LN(4) = 0.03t$	$Log_a(a) = 1$
$[LN(4)]/0.03 = t \approx 46.2$ yrs	Division Property of Equality

So it took a little over 23 years for the population to double. The population doubled again after another 23 years. As mentioned in the previous section, exponential growth results in a doubling at a fixed interval of time.

Example: A car's value is declining exponentially. The car is currently 3 years old and has a value of $18000. The car sold for $26,000 brand new. How much will the car be worth 5 years from now?

Since the new price was $26,000, $N_o = 26000$. This gives us
$N(t) = 26000e^{kt}$

We can use the given data to solve for k by noting that at t=3, N(3) = 18000. This gives us

$18000 = 26000e^{k \bullet 3}$
Now, solve for k.

$18000/26000 = e^{3k}$	Division Property of Equality
$18/26 = e^{3k}$	Simplify Fraction
$LN(18/26) = LN(e^{3k})$	If A = B, LN(A) = LN(B)
$LN(18/26) = 3k \bullet LN(e)$	Power Rule For Logs
$LN(18/26) = 3k$	$Log_a(a) = 1$
$[LN(18/26)]/3 = k \approx -0.1226$	Division Property of Equality

So now, the completed model is $N(t) = 26000e^{-0.1226t}$

We can now use the completed model to answer our question, "How old will the car be in 5 years?" Since the car is already 3 years old, in 5 years it will be 8 years old, so we let t = 8 in our completed model to get

$N(8) = 26000e^{-0.1226 \bullet 8}$
= $9750, rounded

You can verify this answer with the Car Value Calculator, shown at
http://www.mathmotivation.com/science/carvalue.html
You will note that the answer is close, but not an exact match – this is due to the fact that we rounded our value of k more than the calculator does.

General Procedure For Exponential Growth & Decay Applications
If you noticed, in the last problem, we followed this procedure:
1. Use the given data to solve for the constants k and N_o, unless these constants are already given.
2. Write the completed model.
3. Use the completed model to predict other values.

Compound Interest – Solving For t

Now that we are able to solve for variables that are exponents, we may solve the compound interest formula $A = P(1 + r/n)^{nt}$ for t. In other words, we can calculate the time required for a given amount of money P to reach some given future value A.

Example: A person places $4000 in a bank account that earns 5% interest compounded monthly. How many years will it take for the total value to reach $100,000?

We let P = 4000, r=0.05, A = 100000, and n = 12 in $A = P(1 + r/n)^{nt}$ to get

$100000 = 4000(1 + 0.05/12)^{12t}$

Now, solve this.

$100000/4000 = (1 + 0.05/12)^{12t}$	Division Property of Equality
$25 = (1 + 0.05/12)^{12t}$	Simplify Fraction
$LN(25) = LN[(1 + 0.05/12)^{12t}]$	If A = B, LN(A) = LN(B)
$LN(25) = 12t \bullet LN(1 + 0.05/12)$	Power Rule For Logs
$LN(25)/LN(1 + 0.05/12) = 12t$	Division Property of Equality
$[LN(25)/LN(1 + 0.05/12)]/12 = t$	Division Property of Equality
t = 64.5 years (rounded)	Evaluate Logs and divide

You may check this answer by plugging back into the original equation

$100000 = 4000(1 + 0.05/12)^{12t}$

Use of the Log Scales

Base-10 log scales are used in different areas of science. Common applications include the pH scale and the Richter scale. An example involving pH is included here.

Example: pH is commonly used to define acidity in a chemical solution. pH is defined as $pH = -LOG_{10}[H+]$. If pH is 5.2, what is [H+]?

Letting pH = 5.2 results in $5.2 = -LOG_{10}[H+]$. We multiply both sides by –1 to get $-5.2 = LOG_{10}[H+]$

Now, solve this log equation.

$-5.2 = LOG_{10}[H+]$	Given
$10^{-5.2} = [H+]$	Rewrite in equivalent exponential form
[H+] = 0.000006309	Evaluate power of 10
$[H+] = 6.309 \times 10^{-6}$	Rewrite in scientific notation

Did You Know This About pH? The lower the pH, the more acidic a solution is. Since pH uses a base-10 logarithmic scale, it takes a 10-fold increase in acidity to lower the pH 1 unit. With respect to the current issue of “acid rain”, if a lake’s pH decreases by 1 unit, the acidity of a lake increases by a power of 10!