Linear Functions

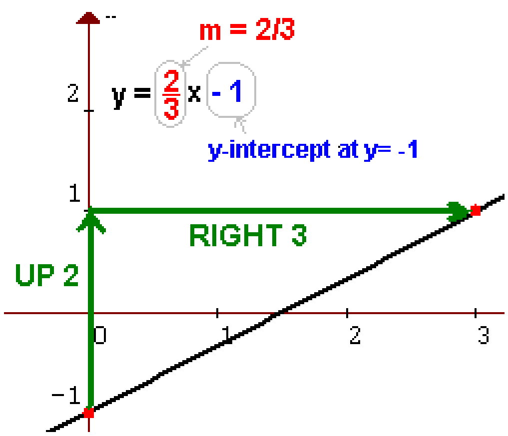
A linear function in two variables is any equation of that may be written in the form y = mx + b where m and b are real number coefficients and x and y represent any real numbers that make up a solution. Furthermore, we observe that

- The point (0, b) will always be the y-intercept.
- The slope of the line will always equal m.
- The slope is defined as **$m = (y_2 - y_1)/(x_2 - x_1)$** for any two points on the line.

We call y = mx + b the Slope-Intercept Form of the linear equation.

Slope
The slope of a line is defined descriptively as *the ratio of how far up you move divided by how far to the right you move, as you move from one point to any other point on the line.*

Example: The graph of y = 2/3x – 1 has a slope of 2/3 and a y-intercept of (0,-1) as shown below.

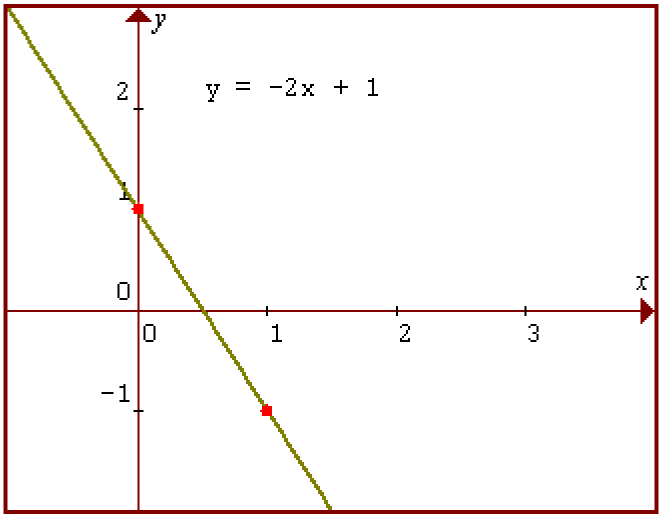


Negative Slope
If we must move down instead of up, a negative sign is part of the slope. Also, if we must move left instead of right, a negative sign is part of the slope. So if slope is negative, we can interpret this in one of two ways:

- m = -(A/B) = (-A)/B which would indicate that you move down A units and then right B units.
- m = -(A/B) = A/(-B) which would indicate that you move up A units and then left B units.

Note that if we must move both down and left when moving from point to point, two negative signs are incorporated into the slope and the result is a positive ratio.

Example: The graph of y = -2x +1 indicates a slope of –2. We may write m = -2 as m = (-2)/1. So we move down 2 and right 1 when moving from point to point as shown below. Alternatively, we could write this slope as m = 2/(-1) and move up 2 and left 1 when moving from point to point.

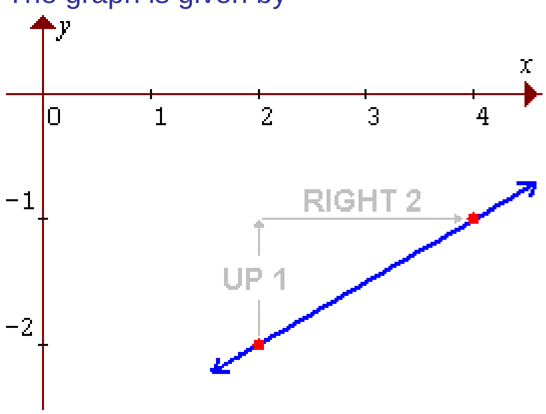


Example: Calculate the slope of the line passing through (4,-1) and (2, -2). Then use this slope to help graph this line.

The slope is given by $\mathbf{m = (y_2 - y_1)/(x_2 - x_1)}$. So we substitute in our values to get

$$m = \frac{-2 - (-1)}{2 - 4} = \frac{-2 + 1}{-2} = \frac{-1}{-2} = \frac{1}{2}$$

The graph is given by



Finding The Equation of a Line: The Point-Slope Form
The equation of a line that passes through (x_1, y_1) and has slope m is given by

$$m(x - x_1) = y - y_1$$

where (x, y) is *any* point that lies on the line and (x_1, y_1) is a specific point given.

Example: Find the equation of the line with slope **m** = **3** that passes through (**4**, **-1**). Then, write this equation in slope-intercept form.

We substitute **m** = **3** and $x_1=4$, $y_1=-1$ into $m(x - x_1) = y - y_1$ to get

$3(x - 4) = y - (-1)$ which simplifies to
$3(x - 4) = y + 1$

Now, to write this in slope-intercept form, we must solve this equation for y.

Begin by multiplying out with the Distributive Property to get
$3x - 12 = y + 1$

Use the Addition Property of Equality to add –1 to both sides to get
$3x - 13 = y$ or $\mathbf{y = 3x - 13}$.

Note: You can quickly check your answer by verifying that the slope of this equation (m=3) matches the given slope (m=3) and also by substituting in the given point (4,-1) into $y = 3x - 13$ to get $-1 = 3 \bullet 4 - 13$ and verify that it is a solution.

Example: Find the equation of the line that passes through the points (4,-1) and (2, -2).

In this case, we must first calculate the slope. Since we are given two points, we can do this using the slope formula.

The slope is given by $m = (y_2 - y_1)/(x_2 - x_1)$. So we substitute in our values to get

$$m = \frac{-2 - (-1)}{2 - 4} = \frac{-2 + 1}{-2} = \frac{-1}{-2} = \frac{1}{2}$$

Now, substitute in m=1/2 and *either* point (4,-1) or (2,-2) into the point-slope form
$m(x - x_1) = y - y_1$.

I will choose (4,-1) to be (x_1, y_1). Why? No reason. Either point will produce the same equation so I simply chose this first point because I had to choose one.

The equation becomes
$$\frac{1}{2}(x - 4) = y - (-1)$$

Applying the Distributive Property and simplifying results in
$(1/2)x - 2 = y + 1$ Continued. . .

Now, apply the Addition Property of Equality to add –1 to both sides to get

$$y = \frac{1}{2}x - 3$$

As a check, both points (4,-1) and (2,-2) should be solutions for this equation. If they are, you have the correct equation since only one linear equation will contain these two points.

Where Does The Point-Slope Form Come From?

The Point-Slope form is a generalization of the slope formula, which states:

$m = (y_2 – y_1)/(x_2 – x_1)$

If we replace the specific point (x_2, y_2) with a general solution point (x, y), we get

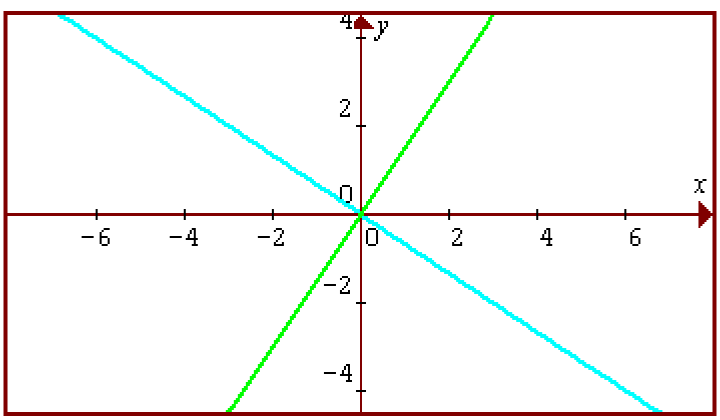
$m = (y – y_1)/(x – x_1)$

Applying the Multiplication Property of Equality to multiply both sides of the equation by $(x – x_1)$ results in $m(x – x_1) = y – y_1$.

Parallel Lines and Perpendicular Lines

Parallel lines are lines with the same slope. If the equations of the lines are in slope-intercept form, we can quickly determine whether or not the lines are parallel. For example, the lines given by $y = 3x – 2$ and $y = 3x + 100$ are parallel since both have a slope of 3.

Perpendicular lines have slopes that are negative reciprocals of each other. For example, y = (3/2)x and y = (-2/3)x are perpendicular since the slopes, 3/2 and –2/3 are negative reciprocals of each other. The graphs of these two functions are shown below.

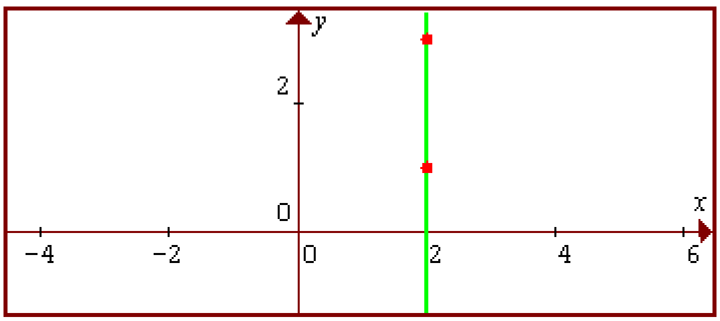


Horizontal and Vertical Lines

Horizontal lines have slope m = 0 and form y = k, where k is the y-value of every point on the horizontal line. The slope is zero because if you move from point to point on any horizontal line, you will move up zero units and right or left some non-zero number of units. This results in a slope of 0 since the numerator of the slope quotient is zero.

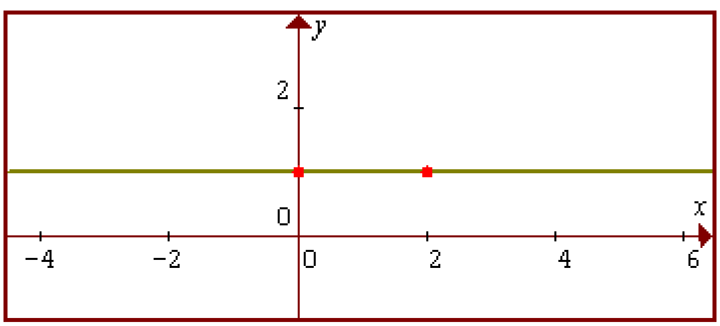
Vertical lines have slope m = undefined and form x = k, where k is the x-value of every point on the vertical line. The slope is undefined because if you move from point to point on any vertical line, you will move up some non-zero number of units and right or left zero units. This results in an undefined slope since the denominator of the slope quotient is zero.

Example: Graph the line through the points (2,1), (2, 3), find its slope, and find its equation.



The slope is undefined, since if you use the slope formula, you get m = (3-1)/(2-2) = 2/0. This is a vertical line with **equation x = 2**, since all points have an x-value of 2.

Example: Graph the line through the points (2,1), (0, 1), find its slope, and find its equation.



The slope is m = (1-1)/(0-2) = 0/(-2) = 0. Or, if you see that it is a horizontal line, you know that the slope is zero. This is a horizontal line with **equation y = 1**, since all points have a y-value of 1.

Putting It All Together
The following 2-part problem requires you to use most of the concepts covered in this section:

A. Find the equation of a line that has graph parallel to the graph of $3x + y = 6$ and passes through the point (2,-1).
B. Also, find the equation of a line that has graph perpendicular to the graph of $3x + y = 6$ and passes through the point (2,-1).

First, we need to realize that two quantities are needed in order to find the equation of a line. We need:

- The slope, or two points so we can calculate the slope.
- One point in addition to the slope.

We can then substitute these values into the point slope form $\mathbf{m(x - x_1) = y - y_1}$.

In this problem, we are given the same point (2,-1) for both parts A and B. So all we need is the slope of the line in each part.

In Part A, our line must be parallel to the line given by $3x + y = 6$. This means it must have the same slope as $3x + y = 6$. To easily find the slope of $3x + y = 6$, rewrite it in slope-intercept form. We get

$y = -3x + 6$ by applying the Addition Property of Equality.

This means the slope is **m=-3**. **Our parallel line in Part A must also have slope m=- 3.**

Substituting m= - 3 and the point (2,-1) into $m(x - x_1) = y - y_1$ results in
$-3(x - 2) = y - (-1)$ or $-3(x - 2) = y + 1$.

We may use the Distributive Property to multiply out to $-3x + 6 = y + 1$ and then use the Addition Property of Equality to add –1 to both sides to get **$y = -3x + 5$ for Part A**.

In Part B, our line must be perpendicular to the line given by $3x + y = 6$. This means it must have a slope that is the negative reciprocal of that of $3x + y = 6$. We rewrote the given line equation as

$y = -3x + 6$

This means the slope of our perpendicular line is **$m = -1/(-3) = 1/3$**.

Substituting m= 1/3 and the point (2,-1) into $m(x - x_1) = y - y_1$ results in
$(1/3)(x - 2) = y - (-1)$ or $1/3(x - 2) = y + 1$.

We may use the Distributive Property to multiply out to $1/3x - 2/3 = y + 1$ and then use the Addition Property of Equality to add –1 to both sides to get **$y = 1/3x - 5/3$ for Part B**.

To check both answers, substitute in x=2, y=-1 into both equations. You should get a solution.