Graphing Rational Functions

A rational function is defined here as a function that is equal to a ratio of two polynomials p(x)/q(x) such that the degree of q(x) is at least 1.

Examples:

$f(x) = \frac{x^2 - 1}{x + 2}$ is a rational function since it is a ratio of two polynomials with degree in the denominator greater than or equal to 1.

$g(x) = \frac{x^3 - x}{2}$ is **not** a rational function since the degree of the denominator is not greater than or equal to 1.

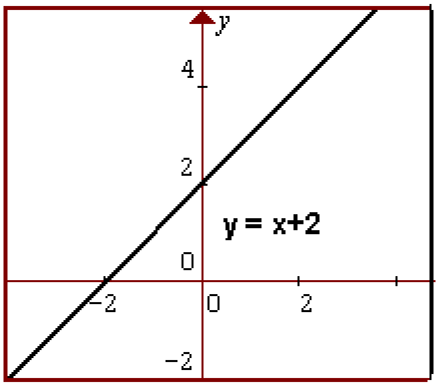
$h(x) = \frac{1 + \sqrt{x}}{x^2 - 3}$ is **not** a rational function since the numerator is not a polynomial.

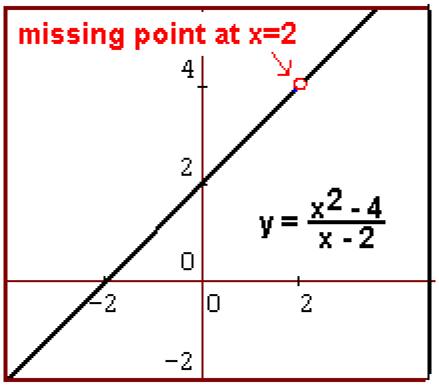
Reduced Rational Functions

A reduced rational function is one where there are no factors common to the numerator and denominator. For example, $y = (x - 1)/(x^2 - 4)$ is in reduced form since there is no factor of (x-1) in the denominator.

Example of Non-Reduced Form: $y = (x^2 - 4)/(x - 2)$ is in non-reduced form since it may be written as

y =[(x + 2)(x-2)]/(x-2)
which may be reduced to
y = x + 2 after canceling common factors of (x-2). So the equation simplifies to a linear equation! However, since x=2 results in division by zero in the original function, we are missing the point at x=2. The graphs of both y=x+2 and our rational function are shown below.

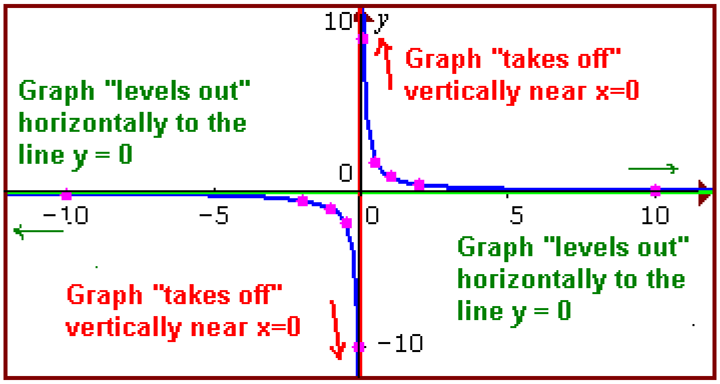




TIP: Always simplify a rational function first, if possible. And remember to exclude any values of x from the domain that result in division by zero.

The Simplest Rational Function: y = 1/x

If you graph the simplest rational function y = 1/x, you get the solution points and graph shown (in blue) below:



x	y
1	1
2	1/2
10	1/10
1/10	10
-1	-1
-2	-1/2
-10	- 0.1
- 0.1	-10

As pointed out, the graph "takes off" vertically for x-values near x=0 and gets closer and closer to the vertical line x=0. We call x=0 the **Vertical Asymptote**.

Also, the graph "levels out" to the horizontal line y=0 for very large positive and negative values of x. We call y=0 the **Horizontal Asymptote**.

Properties of Vertical and Horizontal Asymptotes

- Given a rational function is in reduced form, a vertical asymptote will **always** occur at a value of x that results in division by zero. This is due to the fact that as the denominator gets closer and closer to zero in value, the y-value of the positive or negative value of the function gets larger and larger. Note that for unreduced rational functions, this is not always the case.

- Assuming a rational function has a horizontal asymptote (not all do), the horizontal asymptote may always be approximated by inputting very large positive or negative values of x. If the y-values obtained get closer and closer to some fixed value, then the horizontal asymptote will be given by the horizontal line equal to that value.

Rules For Finding Horizontal Asymptotes

For rational functions where the degree of the denominator is greater than the degree of the numerator, y = 0 will be the horizontal asymptote.

Example $f(x) = \frac{x^{2} - 1}{x^{3} + 2x}$ Has Horizontal Asymptote y = 0

Degree = 2

Degree = 3

For rational functions of the form where the degree of the numerator equals the degree of the denominator, the horizontal asymptote will consist of the horizontal line equal to the ratio of the leading coefficients.

Example $f(x) = \frac{4x^{2} - 1}{5x^{2} + 2x}$ Degree = 2 Degree = 2 Has Horizontal Asymptote $y = \frac{4}{5}$

Slant Asymptotes: The slant asymptote of a *rational function* consists of a slanted line of the standard linear form $y=mx + b$, $m\neq 0$, where the graph of f(x) approaches this linear function as x approaches positive or negative infinity.

The slant asymptote only occurs if the numerator is of degree one more than the degree of the denominator. The slant asymptote may be found by dividing the numerator of the rational function by the denominator and rewriting the rational function as this result. The slant asymptote will be equal to the non-fractional part of this result.

Example $f(x) = \frac{2x^{2} + x + 1}{x + 1}$ has slant asymptote $y = 2x - 1$

$$\begin{array}{rl} & \boxed{2x - 1} + \frac{2}{x+1} \\ x + 1 & \overline{)\,2x^{2} + x + 1} \\ & -\underline{(2x \;\; + 2x)} \\ & -x + 1 \\ & -\underline{(-x \;\; - 1)} \\ & 2 \end{array}$$

Note: The alternative divided out form of f(x) is $f(x) = 2x - 1 + \frac{2}{x+1}$.

This graph approaches the line $y = 2x-1$ for very large positive and negative values of x.

Method For Graphing Rational Functions

1. Make sure the rational function is reduced.
2. Find and plot all asymptotes.
3. Plot several points on each side of each vertical asymptote.
4. Use that fact that the graph “takes off” near each vertical asymptote and “levels out” to each horizontal or slant asymptote to complete the graph.

Example: Graph $y = \frac{x}{x^2 - 4}$

First, make sure this is reduced. The factored form is

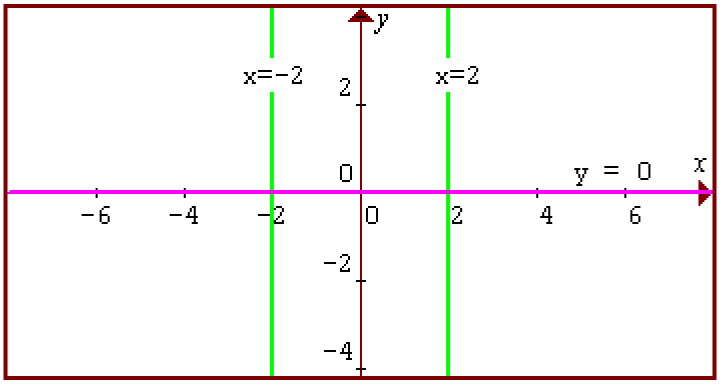
$$y = \frac{x}{(x + 2)(x - 2)}$$

and no factors cancel, so this is the reduced form.

Now, find and plot all asymptotes. The vertical asymptotes will be x=2 and x= -2 since these result in division by zero.

Since the degree of the numerator is 1 and the degree of the denominator is 2, y=0 will be the horizontal asymptote. There is no slant asymptote.

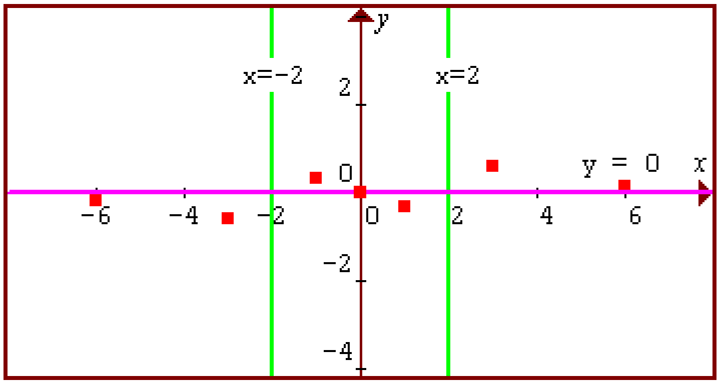
Plot these asymptotes, as shown below.



Now, find and plot a few points on each side of x=2 and x= -2.

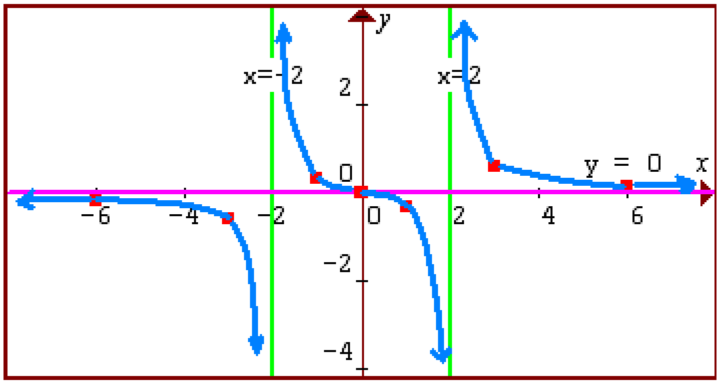
We get the points as shown in the table below by simply inputting values of x into the function and finding the y-values. Then plot these to get

x	y
0	0
1	-1/3
-1	1/3
3	3/5
-3	-3/5
6	6/32 = 3/16
-6	6/-32 = -3/16

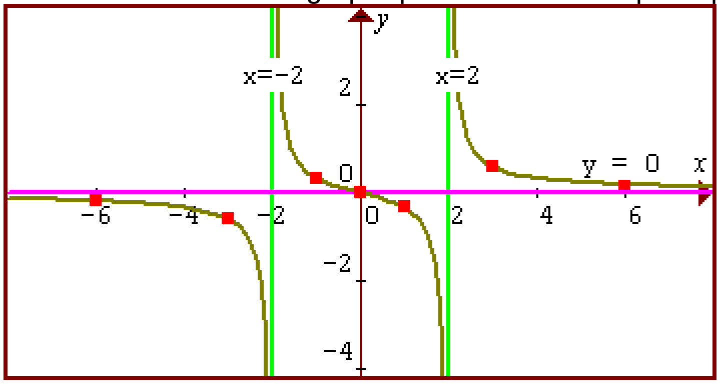


Now, think!

The graph must “take off” near the vertical asymptotes x=2 and x=-2 and “level out” to the horizontal asymptote y=0. Using the points plotted, we can continue the “trends” to get the graph



And here is the “actual graph” plotted with a computer program.



The approximated graph was pretty close to the actual graph!

Example: Plot the graph of $y = x^2/(x-1)$

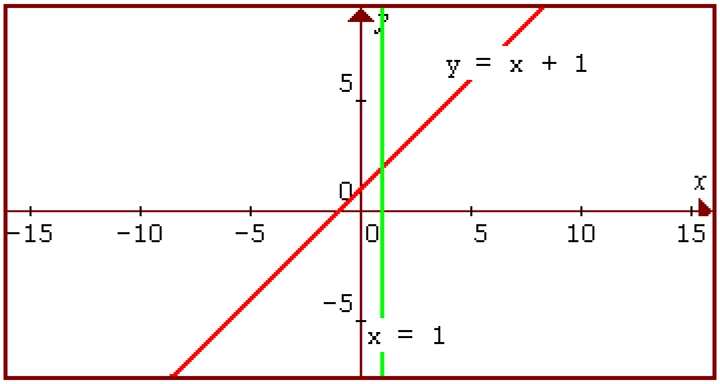
First, is this reduced? Answer: Yes, since nothing can be canceled.

Now, find asymptotes. The vertical asymptote will be x=1 since x=1 results in division by zero. There is no horizontal asymptote since the degree of the numerator is neither equal to or less than the degree of the denominator. And, in this case, since the degree of the numerator is one more than the denominator, we know that there will be a slant asymptote. To find this slant asymptote, we long divide x^2 by (x-1) and using the quotient as our slant asymptote.

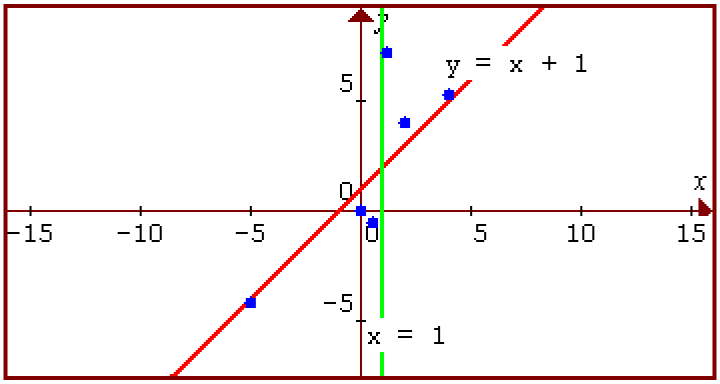
Long division results in a slant asymptote of y = x + 1.

$$\begin{array}{r l}
 & \boxed{y = x + 1} + \frac{1}{x-1} \\
x - 1 & \overline{)\,x^2 + 0x + 0} \\
 & \underline{-(x^2 - x)} \\
 & \quad x + 0 \\
 & \quad \underline{-(x - 1)} \\
 & \qquad 1 = r
\end{array}$$

We now plot both x = 1 and y = x + 1 to get

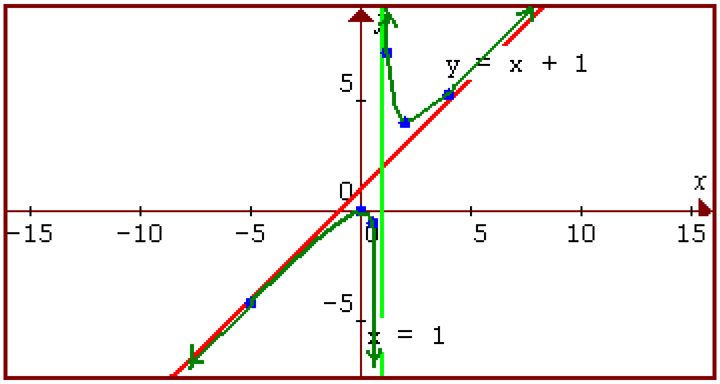


Now, plot several points on each side of x=1 to get

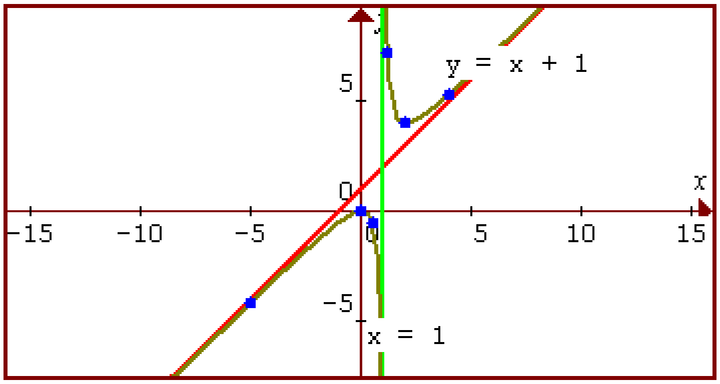


x	y
2	4
1.2	7.2
4	$5\frac{1}{3}$
0	0
0.5	-0.5
-5	$\frac{-25}{6}$

Complete the graph by “leveling out the graph” to the slant asymptotes and having the graph “take off” straight up or down near the vertical asymptote to get



Here is the actual graph, plotted using a computer program:



Again, our method gave us a very good representation!

Why do the rules for horizontal and slant asymptote work?

For rational functions with horizontal asymptote y=0, our numerator has degree less than the denominator and thus increases at a lower rate than the denominator. And as the x-values get larger and larger, the ratio gets closer and closer to zero.

For example, for the function, $y = \frac{x}{x^2 - 2}$
if x = 2, y = 2/2 = 1, but if x=10,
y =10/98 ≈ 0.1.
And if x=100, y = 100/9998 ≈ 0.01 . The denominator grows bigger quicker.

For horizontal asymptotes that are equal to the ratio of leading coefficients, the numerator and denominator have the same degree and thus grow at the same rate. And, if we divide out the numerator by the denominator, we get a constant plus a fraction that approaches zero as x gets large.

For example, for $y = (3x^2 - 1)/(2x^2 + 1)$, long division results in

$$\begin{array}{r l}
 & 3/2 + \boxed{\frac{-3/2x \; - 1}{2x^2 + 1}} \\
2x^2 + 1 & \overline{)3x^2 + 0x \; - \; 1} \\
 & -(3x^2 + 3/2x) \\
\hline
 & -3/2x \; - 1
\end{array}$$

This fraction part approaches zero

So for large values of x, y ≈ 3/2.

The same rational applies to slant asymptotes. The remainder, upon long dividing, approaches zero for large values of x, leaving the linear function.