Exponential and Log Equations

An exponential or log equation is defined her as any equation that contains one or more exponential or logarithmic terms. We will use the property that allows us to take the log of both sides of an equation shown immediately below. Also, in some cases we will solve equations by doing what is often described as "unlogging" both sides of an equation, also covered in this property.

Logarithms of Both Sides of an Equation
Given two positive quantities, U and V, and a legitimate base *a*,
If U = V, Then $LOG_a(U) = LOG_a(V)$.
and likewise
If $LOG_a(U) = LOG_a(V)$, then U = V.

Mathematically, we can write this as *U = V if-and-only-if* $LOG_a(U) = LOG_a(V)$.

Solving Exponential Equations:
To solve an equation containing exponential terms, use the following procedure:

1. Isolate the exponential term on one side of the equal sign.
2. Take the log function of both sides. Use base-10 or base-e logs and match the base of your exponential if possible.
3. Apply the Power Rule for Logs to move the exponent in front of the log.
4. Solve for your variable.
5. Check answers. Sometimes you will get "extra" answers.

Example: Solve $\frac{4}{3e^x} + 1 = 7$ **and round answer to 3 places.**

First, you must solve for the exponential term e^x using a bit of algebra.

$\frac{4}{3e^x} = 6$	Apply the Addition Property of Equality
$4 = 18e^x$	Apply Multiplication Property of Equality
$4/18 = e^x$	Apply Division Property of Equality
$LN(4/18) = LN(e^x)$	Take Logs of Both Sides
$LN(4/18) = x \bullet (LN\ e)$	Apply Power Rule For Logs
$LN(4/18) = x \bullet 1$	Apply Property: $Log_a a = 1$
x = LN(4/18) x = -1.504 (rounded)	Multiplicative Identity: 1x = x

Now, check by inputting x = -1.504 into the original equation. You get

$$\frac{4}{3e^{-1.504}} + 1 = 7$$

which ends up giving you 6.9995 ≈ 7. Close enough!

Equations Containing Logs on One Side of Equals

Equations with one or more log expressions on one side of the equals sign, are solved easiest using the following procedure.

1. If there is more than one log on one side, combine logs using the Log Product Rule or Log Quotient Rule.
2. Once the equation is in the form U = $Log_a x$, rewrite in exponential form as $a^U = x$. Then solve for x.
3. Check answers. You will *often* get extra non-working solutions.

Example: Solve $LOG_{10}(x) - LOG_{10}(x - 1) = 2$ and round to 4 places

First, combine the logs on the left using the Log Quotient rule to get

$$LOG_{10}\left(\frac{x}{x-1}\right) = 2$$

Now, write in equivalent exponential form to get

$$\frac{x}{x-1} = 10^2 = 100$$

Now, solve for x.

x = 100(x – 1)	Using the Multiplication Property of Equality
x = 100x – 100	Apply the Distributive Property
100 = 99x	Use the Addition Property of Equality (twice)
$100/99 = x \approx 1.0101$	Use the Division Property of Equality

Now, check x = 1.0101 in the original equation $LOG_{10}(x) - LOG_{10}(x - 1) = 2$. We get

$LOG_{10}(1.0101) - LOG_{10}(1.0101 - 1) = 2$

$LOG_{10}(1.0101) - LOG_{10}(0.0101) = 2$

$2.00004 \approx 2$ Close enough, considering that we rounded.

Equations Containing Logs on Both Sides of Equals

Equations with one or more log expressions on both sides of the equals sign, are solved easiest using the following procedure.

1. If there is more than one log on one side, combine logs using the Log Product Rule or Log Quotient Rule.
2. Once the equation is in the form $Log_a U = Log_a V$, rewrite as U = V. Then solve for your variable.
3. Check answers. You will *often* get extra non-working solutions.

Example: Solve LN(x + 2) + LN x = LN(x – 1)

First, consolidate logs on the left by applying the Log Product Rule to get
LN[(x + 2)(x)] = LN(x + 1)

Now, you can "unlog" both sides to get
(x + 2)(x) = x + 1

Now, solve for x.

$x^2 + 2x = x + 1$	Apply the Distributive Property
$x^2 + x - 1 = 0$	Move terms to one side using Addition Property of Equality

Solve with the Quadratic Formula to get

$$x = \frac{-1 \pm \sqrt{5}}{2}$$

In decimal form, this results in $x = (-1 + \sqrt{5})/2 \approx 0.618$ and $x = (-1 - \sqrt{5})/2 \approx -1.618$
Now, check the two results in LN(x + 2) + LN x = LN(x – 1) .

LN(0.618 + 2) + LN(0.618) = LN(0.618 – 1)
This does **NOT** check since the right side can not be evaluated!

LN(-1.618 + 2) + LN(-1.618) = LN(-1.618 – 1)
This also does **NOT** check since neither right side nor left can be evaluated!
Answer: NO SOLUTION

Why Does "Unlogging" Work?

Many students mistakenly think that the "unlogging" process is the equivalent of canceling out a log of each side.

$$\cancel{LOG_{10}}\, x^2 = \cancel{LOG_{10}}\,(x - 1) \quad \longleftarrow \text{WRONG!}$$

What you are *actually* doing is inputting both sides into the inverse function of the log. In this case, you would input each side into the 10^X function. You get

$$10^{LOG_{10} x^2} = 10^{LOG_{10}(x - 1)} \quad \longleftarrow \text{RIGHT!}$$

$$x^2 = x - 1$$

So we are doing the composition of $f(x) = 10^X$ with its inverse $f^{-1}(x) = LOG_{10}X$, which results in the input value. In other words, we are applying the **Inverse Property of Logs** to get $x^2 = x-1$.