Exponential Functions

Exponential functions are perhaps the most important class of functions in mathematics. We use this type of function to calculate interest on investments, growth and decline rates of populations, forensics investigations, as well as in many other applications.

Definition of an Exponential Function

An exponential function has the form:

$f(x) = a^x$

where "a" is the base, a > 0, and a is not 1.
x is any real number.
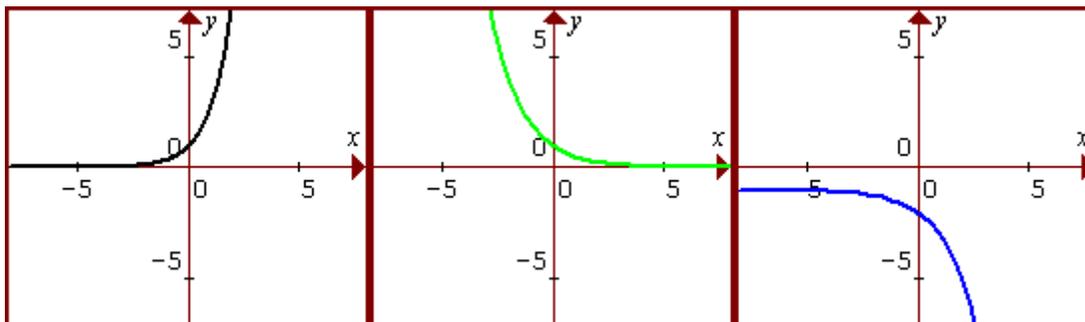
Examples:

$f(x) = 2^x$, $g(x) = 3^x$, $y = (1/2)^x$, $y = (0.1)^x$
are all exponential functions.

Also, compositions of an exponential function with another function are also referred to as *exponential*. An example would be $f(x) = 4 + 100 \cdot 3^{-2x}$.

Graphs of Exponential Functions

Below are the graphs of 3 different exponential functions. What do you notice?


Exponential graphs share these common features:

- The graph will level out on the far right or the far left to some horizontal asymptote.
- The graph "takes off" vertically, but it does not approach a vertical asymptote. Rather, it simply becomes steeper and steeper.
- The graph will have a characteristic "L" shape, if you zoom out enough.

Example: Graph $y = 2^x$. Then use function shift rules to graph $y = 1 + 2^x$, $y = 2^{(x-3)}$, and $y = 2^{-x}$.

If you simply calculate and plot some points for $y = 2^x$, you see that the graph levels out to the horizontal axis and takes off vertically fairly quickly as shown below.
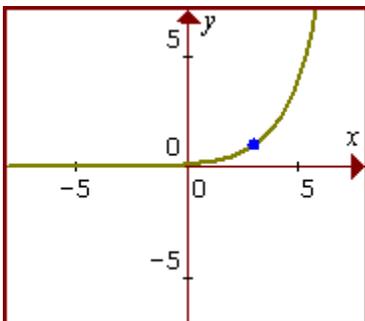
| x	0	1	2	3	-1	-2
y	1	2	4	8	½	¼
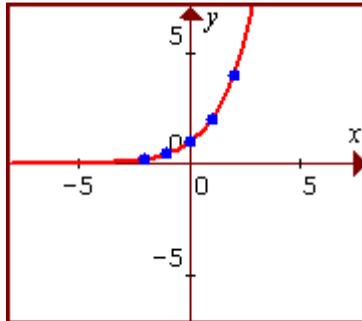
Now, to graph $y = 1 + 2^x$, shift the graph up 1 unit and you get the graph shown below. Notice that the graph levels out to the horizontal asymptote y=1 instead of y=0. Also, the y-intercept (0,1) has been shifted up 1 to (0,2).
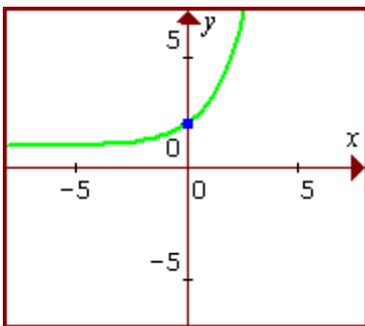
To graph $y = 2^{(x-3)}$, shift the graph right 3 unit and you get the graph shown below. The y-intercept (0,1) has been shifted right 3 to (3,1).

To graph y = 2^{-x} , reflect the graph of y = 2^x across the y-axis as shown below.
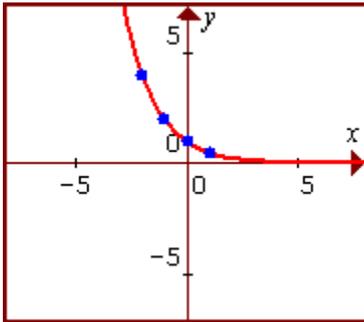
This graph, like y=2^x, levels out to the horizontal asymptote y=0, except on the right side instead of the left.

TIP: To graph exponential functions, you only need to find enough points to generate the "L" shape of the graph. Also, use function shift rules if applicable to save a lot of time.

The Most Common Exponential Base: e
Many students assume that either 10 or 2 is the most common base, since those are the bases we use. But, actually, the most common base is *e*, where

*e* = 2.71828182846. . . , an irrational number.

Also, *e* is defined exactly as $e = (1 + 1/m)^m$ as m increases to infinity. You can see how this definition produces *e* by inputting a large value of m like m=10,000,000 to get

(1 + 1/10000000)10000000 = 2.7182817 (rounded), which is very close to the actual value.

Example: Graph y = e^x

To graph this, you would input values much like you did to graph y=2X. The difference here is that you will have to use a scientific calculator to find the function values. You will need to use your e^x function, which normally requires use of the 2nd function key.

X	0	1	2	3	-1	-2
Y	1	2.718	7.389	20.086	0.368	0.135
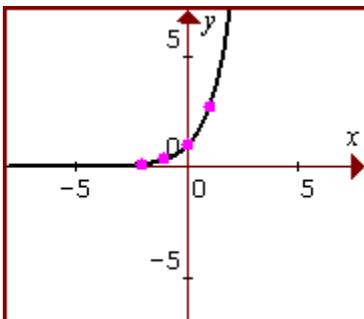
As you can see, the graph of $y=e^X$ is very similar to $y=2^X$. The only difference is that the graph levels off to y=0 a bit quicker and it gets vertically steeper quicker.

TIP: When graphing $y = a^X$, the value of "a" determines how quickly the graph levels out and takes off vertically. Otherwise, all of the graphs of this form will level out to y=0 and take off vertically forming an "L" shape.

Application of Exponentials: Compound Interest

When an amount of money, "P", is invested at an annual rate of interest, "r" (given as a decimal), for a total of "t" years, compounded "n" times per year, the total amount of money "A" is

$$A = P(1 + r/n)^{nt}$$

Example: You place $4000 in a bank account that earns 5% interest compounded quarterly. What is the total amount in the account after 3 years?

In this case, P = 4000, t=3, and r = 0.05 . Since interest is compounded quarterly, there are 4 compounding periods per year and n=4. Substituting in these values results in

$$A = 4000(1 + 0.05/4)^{4 \bullet 3}$$
$$= 4000(1.160754518)$$
$$= \$4643.02$$

TIP: To calculate this quickly and accurately, enter 1 + 0.05 ÷ 4 = $\boxed{x^y}$ 12 = on your calculator. Then multiply this answer by 4000. Round off only after you get the final answer.

Continuous Compounding And "e"

The more often money is compounded, the more money it makes. For example, in the previous example, if interest were compounded daily instead of quarterly, n would have equaled 365 and the total amount would have been

$$A = 4000(1 + 0.05/365)^{365 \bullet 3}$$
$$= 4000(1.160754518)$$
$$= \$4647.29$$

If we compound interest instantly, i.e. keep increasing the number of compounding periods per year to infinity, we would end up with what is called continuous compounding of interest.

The formula for continuously compounded interest is
$$A = Pe^{rt}$$

Where P is the original principal, r is the annual rate, t equals the number of years, and e is the exponential base e = 2.71828182846. ..

For the previous example where $4000 was invested at 5% interest for 3 years, continuous compounding would result in

$A = 4000(e^{0.05 \bullet 3})$
    = $4647.34 , which surprisingly, is only 5 cents more than daily compounding.

Population Growth, Exponential Growth, and "e"

Populations, plants, and interest compounded continuously all experience "continuous growth" the same way in that there is growth occurring each instant. After all, babies are not born all on a certain day of the month, nor do your bean plants sit with no growth until a certain day and then grow suddenly!

So for *any* quantity that grows exponentially, $\mathbf{A = Pe^{rt}}$ where
P = the original amount at time t=0.
r = the (decimal) rate of growth per time period, be it years, days, etc.
t = the units used in the time period.

Example: A plant's mass initially increases exponentially at a rate of 5% per day. On the initial day of recorded growth, the plant has mass of 10g. What is the mass of the plant after 4 days and then again after 15 days and 30 days from the initial day?

This plant will grow according to the formula
$A = Pe^{rt}$ where
P = 10, r = 0.05, and t = 4 and t=30.

We get
$A = 10e^{0.05 \bullet 4}$
    = 12.2 g after 4 days

and

$A = 10e^{0.05 \bullet 15}$
    = 21.1 g after 15 days

and

$A = 10e^{0.05 \bullet 30}$
    = 44.8 g after 30 days

Notice that the mass approximately doubles every 15 days**? In general, if something is growing exponentially, it will double in size at a regular time interval.** This also applies to money invested in a bank earning interest – the total amount will double at a regular time interval.